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A study of semiquasihomogeneous singularities by using holonomic system (holonomic 系を用いた半擬斉次孤立特異点の考察)

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1 Introduction

In this note, we study the quasihomogeneous singularity and exceptional singularities of modality 1 by using partial differential operators. We examine, in particular, algebraic local cohomology classes attached to these singularities.

Let X be an open neighborhood of the origin O in the n dimensional affine space \mathbb{C}^n . Let f be a holomorphic function with an isolated singularity at the origin O . Denote by I the ideal in the sheaf \mathcal{O}_X of holomorphic functions on X generated by the partial derivatives $f_j = \frac{\partial f}{\partial z_j}$ ($j = 1, \dots, n$) of the function f . We denote by Σ the set of cohomology classes in $\mathcal{H}_{[0]}^n(\mathcal{O}_X)$ annihilated by every element in I . Since the pairing

$$\Omega_{X,O}/I\Omega_{X,O} \times \Sigma \rightarrow \mathbb{C} \quad (1.1)$$

defined by the Grothendieck local residue is non-degenerate, Σ becomes the dual space of $\mathcal{O}_{X,O}/I \cong \Omega_{X,O}/I\Omega_{X,O}$ as a vector space where Ω_X is the sheaf of holomorphic differential n -forms on X .

Let σ be an algebraic local cohomology class which generates Σ over $\mathcal{O}_{X,O}$. Since the algebraic local cohomology group $\mathcal{H}_{[0]}^n(\mathcal{O}_X)$ has a structure of \mathcal{D}_X modules, we can consider annihilators of σ in the sheaf \mathcal{D}_X of differential operators on X . In this paper, we consider the ideal, denoted by $\text{Ann}_{\leq 1}$, in \mathcal{D}_X generated by annihilators of σ of at most first order.

We give the precise definition of $\text{Ann}_{\leq 1}$ and we give an description of the solution space of the holonomic system $\mathcal{D}_X/\text{Ann}_{\leq 1}$ in §2.

In §3, we examine $\text{Ann}_{\leq 1}$ in the case of quasihomogeneous singularities. We verify that the cohomology class σ attached to a quasihomogeneous singularity can be characterized as the solution of the system of differential equations of at most first order (Theorem 3.1).

For non-quasihomogeneous isolated singularities, we show that the dimension of the solution space of the holonomic system $\mathcal{D}_X/\text{Ann}_{\leq 1}$ is greater than or equal to 2. Especially, in the case of exceptional singularities of modality 1, we verify that the dimension of the solution space of $\mathcal{D}_X/\text{Ann}_{\leq 1}$ is just equal to 2 and the basis is given by σ and the delta function (Theorem 4.1 in §4). In §4.4, we give results of computations for normal forms of exceptional singularities of modality 1.

2 The first order differential operators acting on Σ

For a holomorphic function $f = f(z_1, \dots, z_n)$ with an isolated singularity at the origin O , let I be the ideal in $\mathcal{O}_{X,O}$ generated by the partial derivatives $f_j = \frac{\partial f}{\partial z_j}$ ($j = 1, \dots, n$):

$$I = \langle f_1, \dots, f_n \rangle_O.$$

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Denote by Σ the set of local cohomology classes annihilated by every element in I :

$$\Sigma = \{\eta \in \mathcal{H}_{[O]}^n(\mathcal{O}_X) \mid g\eta = 0, g \in I\}.$$

Let σ be a generator of Σ over $\mathcal{O}_{X,O}$:

$$\Sigma = \mathcal{O}_{X,O}\sigma.$$

Let P be a partial differential operator of first order which annihilates the algebraic local cohomology class σ . Such an operator P has the following property.

Lemma 2.1 *Let σ be a generator of Σ over $\mathcal{O}_{X,O}$. Let P be a linear partial differential operator of first order such that $P\sigma = 0$. Then, we have $P(\Sigma) \subseteq \Sigma$.*

Proof. Since σ generates Σ over $\mathcal{O}_{X,O}$, we can write any $\eta \in \Sigma$ as $\eta = h\sigma$ with some holomorphic function $h \in \mathcal{O}_{X,O}$. Let v_P be the first order part $\sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j}$ of the annihilator $P = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j} + a_0(z)$. We have

$$\begin{aligned} P(\eta) &= P(h\sigma) \\ &= (Ph - hP)\sigma + hP\sigma \\ &= v_P(h)\sigma \in \Sigma. \end{aligned}$$

Thus we have $P(\Sigma) \subseteq \Sigma$. \square

Let \mathcal{L} be the set of linear partial differential operators of at most first order which annihilate σ :

$$\mathcal{L} = \{P = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0\}.$$

It is obvious from the proof of Lemma 2.1, the condition whether a given first order differential operator R acts on Σ or not depends only on the first order part v_R of R . We denote by \mathcal{V} the set of differential operators of the form $\sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j}$ acting on Σ . Then, $v = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j}$ is in \mathcal{V} if and only if v satisfies the condition $vg \in I$ for any $g \in I$, i.e.,

$$\mathcal{V} = \{v = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j} \mid vg \in I, \forall g \in I\}.$$

Lemma 2.2 *The mapping, from \mathcal{L} to \mathcal{V} , which associates the first order part $v_P \in \mathcal{V}$ to $P \in \mathcal{L}$ is a surjective mapping.*

Proof. For any $v \in \mathcal{V}$, there exists a holomorphic function $h \in \mathcal{O}_{X,O}$ such that $v\sigma = h\sigma$. Then we have an annihilator $P = v - h \in \mathcal{L}$. \square

Let us consider the condition that the class $\eta \in \Sigma$ becomes a solution of homogeneous differential equation $P\eta = 0$ for an annihilator P of $\sigma \in \Sigma$. There exists a holomorphic function $h \in \mathcal{O}_{X,O}$ which satisfies $\eta = h\sigma$. Then we have

$$v_P h = \sum_{j=1}^n a_j(z) \frac{\partial h}{\partial z_j} \in I$$

where $v_P \in \mathcal{V}$ is the first order part of the differential operator P . It is obvious that to represent $\eta \in \Sigma$ in the form $\eta = h\sigma$, it suffices to take the modulo class $h \bmod I$ of the holomorphic function $h \in \mathcal{O}_{X,O}$. An element $v \in \mathcal{V}$ induces a linear operator acting on $\mathcal{O}_{X,O}/I$ which is also denoted by v :

$$v : \mathcal{O}_{X,O}/I \rightarrow \mathcal{O}_{X,O}/I.$$

We can put

$$\mathcal{H} = \{h \in \mathcal{O}_{X,O}/I \mid vh = 0, \forall v \in \mathcal{V}\}.$$

Put $\text{Ann}_{\leq 1} = \mathcal{D}_X \mathcal{L}$. $\text{Ann}_{\leq 1}$ defines a left ideal in \mathcal{D}_X . We have the next theorem.

Theorem 2.1

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) = \text{Span}\{h\sigma \mid h \in \mathcal{H}\}.$$

Proof. Since $I \subset \text{Ann}_{\leq 1}$ as an ideal of multiplicative operators, we have

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) \subset \Sigma.$$

Thus we can write any solution of the holonomic system $\mathcal{D}_X/\text{Ann}_{\leq 1}$ as $h\sigma$ for some $h \in \mathcal{O}_{X,O}$. For $P \in \mathcal{L}$, we have

$$P(h\sigma) = v_P(h)\sigma = 0.$$

Thus we have $v_P h = 0$. \square

3 The case of quasihomogeneous singularities

Let σ be a generator of Σ over $\mathcal{O}_{X,O}$. Let Ann be a left ideal in \mathcal{D}_X consisting of annihilators of the algebraic local cohomology class σ .

Theorem 3.1 *The following three conditions are equivalent :*

- (i) $\mathcal{O}_{X,O}\langle f, f_1, \dots, f_n \rangle = \mathcal{O}_{X,O}\langle f_1, \dots, f_n \rangle$, where $f_j := \frac{\partial f}{\partial z_j}$, $j = 1, \dots, n$.
- (ii) $\text{Ann}_{\leq 1} = \text{Ann}$.
- (iii) $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) = \text{Span}\{\sigma\}$.

Proof.

(i) \Rightarrow (ii) : Suppose that $\mathcal{O}_{X,O}\langle f, f_1, \dots, f_n \rangle = \mathcal{O}_{X,O}\langle f_1, \dots, f_n \rangle$ for $f \in \mathcal{O}_{X,O}$. Then the function f can be expressed in terms of the derivatives f_1, \dots, f_n . We have

$$\begin{aligned} f &= a_1 f_1 + \dots + a_n f_n \\ &= a_1 \frac{\partial f}{\partial z_1} + \dots + a_n \frac{\partial f}{\partial z_n} \\ &= (a_1 \frac{\partial}{\partial z_1} + \dots + a_n \frac{\partial}{\partial z_n})f, \end{aligned} \tag{3.1}$$

with $a_1, \dots, a_n \in \mathcal{O}_{X,O}$. Assume that $(a_1, \dots, a_n) \neq (0, \dots, 0)$. Put $v = a_1 \frac{\partial}{\partial z_1} + \dots + a_n \frac{\partial}{\partial z_n}$. From (3.1), we have $f_j = (a_{1j} f_1 + \dots + a_{nj} f_n) + v f_j$ where $a_{kj} = \frac{\partial a_k}{\partial z_j}$. As $v f_j = f_j - (a_{1j} f_1 + \dots + a_{nj} f_n) \in I$, we have $v \in \mathcal{V}$. From Lemma 2.2, we have an annihilator $P = v + a_0$ of the cohomology class σ for some $a_0 \in \mathcal{O}_{X,O}$. We have

$$\langle f_1, \dots, f_n, P \rangle \subseteq \text{Ann}_{\leq 1} \subseteq \text{Ann}.$$

It is known in [2] that the Jacobian of a_1, \dots, a_n is not zero at the origin. This assures that the holonomic system $\mathcal{D}_X/\langle f_1, \dots, f_n, P \rangle$ becomes simple. Since the holonomic system \mathcal{D}_X/Ann is simple, we have

$$\langle f_1, \dots, f_n, P \rangle = \text{Ann}_{\leq 1} = \text{Ann}.$$

(iii) \Rightarrow (i) : Assume that $f \notin \mathcal{O}_{X,O}\langle f_1, \dots, f_n \rangle$. Obviously, we have $f\sigma \neq 0$. Let us denote by $F \in \mathcal{D}_X$ the multiplicative operator defined by $f \in \mathcal{O}_{X,O}$. If the differential operator $P = \sum_{j=1}^n a_j \frac{\partial}{\partial z_j} + a_0$ annihilates the cohomology class σ , we have

$$\begin{aligned} P(f\sigma) &= PF\sigma \\ &= (PF - FP)\sigma + FP\sigma \\ &= \sum_{j=1}^n a_j \frac{\partial f}{\partial z_j} \sigma. \end{aligned}$$

Since $\sum_{j=1}^n a_j f_j \in I$,

$$P(f\sigma) = 0$$

holds. Thus, there exist at least 2 elements σ and $f\sigma$ in $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X))$. As σ and $f\sigma$ are linearly independent elements, we have

$$\dim \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) \geq 2.$$

(ii) \Rightarrow (iii) : By assumption, we have

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) = \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}, \mathcal{H}_{[O]}^n(\mathcal{O}_X))$$

where $\text{Ann} = \{P \in \mathcal{D}_X \mid P\sigma = 0\}$. Since \mathcal{D}_X/Ann is simple holonomic system, we have

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) = \text{Span}\{\sigma\}.$$

□

For the holomorphic function f with an isolated singularity at the origin O , suppose that

$$\mathcal{O}_{X,O}\langle f, f_1, \dots, f_n \rangle = \mathcal{O}_{X,O}\langle f_1, \dots, f_n \rangle. \quad (3.2)$$

It is known in [2] there exists some holomorphic coordinate transformation which makes f a quasihomogeneous polynomial. Theorem 3.1 asserts that, it is possible to characterize the algebraic local cohomology class σ attached to a quasihomogeneous singularity as the solution of the system of differential equations of at most first order.

4 The case of exceptional families of singularities of modality 1.

In this Section, we characterize cohomology classes attached to exceptional families of unimodal singularities. Functions having non-degenerate quasihomogeneous principal part of modality 1 can be reduced to three one-parameter families of parabolic singularities and 14 polynomials generating exceptional families. Since the parabolic singularities satisfy (3.2), our objects are the following 14 polynomials.

2variables

$$\begin{aligned} E_{12} &: f(x, y) = x^3 + y^7 + axy^5 \\ E_{13} &: f(x, y) = x^3 + xy^5 + ay^8 \\ E_{14} &: f(x, y) = x^3 + y^8 + axy^6 \\ Z_{11} &: f(x, y) = x^3y + y^5 + axy^4 \\ Z_{12} &: f(x, y) = x^3y + xy^4 + ay^6 \\ Z_{13} &: f(x, y) = x^3y + y^6 + axy^5 \\ W_{12} &: f(x, y) = x^4 + y^5 + ax^2y^3 \\ W_{13} &: f(x, y) = x^4 + xy^4 + ay^6 \end{aligned}$$

3variables

$$\begin{aligned} Q_{10} &: f(x, y, z) = x^3 + y^4 + yz^2 + axy^3 \\ Q_{11} &: f(x, y, z) = x^3 + y^2z + xz^3 + az^5 \\ Q_{12} &: f(x, y, z) = x^3 + y^5 + yz^2 + axy^4 \\ S_{11} &: f(x, y, z) = x^4 + y^2z + xz^2 + ay^2x^2 \\ S_{12} &: f(x, y, z) = x^2y + y^2z + xz^3 + az^5 \\ U_{12} &: f(x, y, z) = x^3 + y^3 + z^4 + axyz^2 \end{aligned}$$

These normal forms of quasihomogeneous singularities are given by V.I.Arnold ([1]).

4.1 The quasidegrees of cohomology classes

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a type of quasihomogeneous singularities. A cohomology class $\eta \in \Sigma$ has an expression

$$\eta = \left[\sum_{\mathbf{k} \in E} c_{\mathbf{k}} \frac{1}{z^{\mathbf{k}}} \right]$$

where $c_{\mathbf{k}} \in \mathbb{Q}$ and $z^{\mathbf{k}} = z_1^{k_1} \dots z_n^{k_n}$ with $\mathbf{k} = (k_1, \dots, k_n)$ and E is a finite subset of \mathbb{N}^n .

Definition 4.1 A cohomology class $\left[\frac{1}{z^{\mathbf{k}}} \right]$ has degree $-d$ if

$$\langle \alpha, \mathbf{k} \rangle = \alpha_1 k_1 + \dots + \alpha_n k_n = d$$

For a cohomology class $\eta = \left[\sum_{\mathbf{k} \in E_\eta} c_{\mathbf{k}} \frac{1}{z^{\mathbf{k}}} \right]$, we define its degree $d(\eta)$ by the smallest degree of classes $\left[\frac{1}{z^{\mathbf{k}}} \right]$ in η :

$$d(\eta) = \min\{-\langle \alpha, \mathbf{k} \rangle \mid \mathbf{k} \in E_\eta\},$$

where E_η is a set of all exponents $\mathbf{k} = (k_1, \dots, k_n)$ of non-zero term $c_{\mathbf{k}} \frac{1}{z^{\mathbf{k}}}$ in the above expression of the cohomology class η . For both functions and cohomology classes, we denote its degree by $d(\cdot)$.

In the case of semiquasihomogeneous singularities, we have the following result.

Proposition 4.1 *Let f be a semiquasihomogeneous function. For any basis monomial m_j of the vector space $\mathcal{O}_{X,O}/I$, there exists a cohomology class η in the vector space Σ which satisfies following two conditions :*

- (i) $m_j \eta = \delta$, where δ is the delta function with support at the origin.
- (ii) $d(\eta) = -\sum_{j=1}^n \alpha_j - d(m_j)$.

Furthermore, we have the following proposition.

Proposition 4.2 *Let f be a semiquasihomogeneous function. A necessary and sufficient condition for a cohomology class $\sigma \in \Sigma$ to be a generator of Σ over $\mathcal{O}_{X,O}$ is*

$$d(\sigma) = -nd(f) + \sum_{j=1}^n \alpha_j.$$

4.2 Cohomology classes attached to exceptional singularities of modality 1.

Recall that, for a non-quasihomogeneous function f , we have

$$\mathcal{O}_{X,O} \langle f_1, \dots, f_n \rangle \neq \mathcal{O}_{X,O} \langle f, f_1, \dots, f_n \rangle \quad (4.1)$$

and thus

$$\dim \text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X / \text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) \geq 2.$$

We examine the solution space $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X / \text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X))$ of the holonomic system $\mathcal{D}_X / \text{Ann}_{\leq 1}$ attached to exceptional singularities of modality 1.

We verify that \mathcal{H} is spanned by 1 and the modulo class of $f(z)$ in $\mathcal{O}_{X,O}/I$. That is, we have the following proposition.

Proposition 4.3 *For a function f defining an exceptional singularity of modality 1, we have*

$$\mathcal{H} = \text{Span}\{1, f \bmod I\}.$$

Proposition 4.3 is proved by direct computations for each normal form of an exceptional family of singularities of modality 1. Since $z_j f \in I$ ($j = 1, \dots, n$), we have $f \bmod I = c_0 j_F(z) \bmod I$ where $j_F(z)$ is the Jacobian $\frac{\partial(f_1, \dots, f_n)}{\partial(z_1, \dots, z_n)}$ and c_0 is a non-zero constant. Thus we have the following theorem.

Theorem 4.1 *Let f be a function defining an exceptional singularity of modality 1. Then, we have*

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X / \text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) = \text{Span}\{\sigma, \delta\},$$

where δ is the delta function with support at the origin O .

To give effects of computations, we introduce the following vector spaces :

$$L = \{P = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j} + a_0(z) \mid P\sigma = 0, a_j(z) \in \mathcal{O}_{X,O}/I, j = 0, \dots, n\},$$

$$V = \{v = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial z_j} \mid v g \in I, \forall g \in I, a_j(z) \in \mathcal{O}_{X,O}/I, j = 1, \dots, n\},$$

$$H = \{h \in \mathcal{O}_{X,O}/I \mid v h = 0, \forall v \in V\}.$$

Lemma 4.1 *We have the isomorphism between L and V :*

$$L \cong V.$$

Proof. For any $v \in V$, there exist $h \in \mathcal{O}_{X,O}$ s.t., $v\sigma = h\sigma$. By putting $a_0 = -h \bmod I$, we have $(v + a_0)\sigma = 0$. \square

4.3 Example : E_{12} singularity.

The quasihomogeneous part of the function $f = x^3 + y^7 + axy^5$ is of type $(7, 3)$ of degree 21. The partial derivatives of f with respect to the variables x and y are $f_x = 3x^2 + ay^5$ and $f_y = 7y^6 + 5axy^4$, respectively. We use the lexicographic order with $x \succ y$ in computations. The standard base of the ideal $I = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle_O$ in $\mathcal{O}_{X,O}$ is

$$\{y^8, 7y^6 + 5ay^4x, 3x^2 + ay^5\}.$$

Basis monomials of the local ring $\mathcal{O}_{X,O}/I$ is given by

$$\begin{array}{cccccccccccccccc} 1, & y, & y^2, & x, & y^3, & yx, & y^4, & y^2x, & y^5, & y^3x, & y^4x, & y^5x \\ 0, & 3, & 6, & 7, & 9, & 10, & 12, & 13, & 15, & 16, & 19, & 22 \end{array}$$

The numbers below the monomials are their quasi-degrees.

Any element of Σ is given by linear combination of the next 12 cohomology classes :

$$\begin{aligned} & \left[\frac{1}{y^6x^2} + a\left(-\frac{5}{7}\frac{1}{y^8x} - \frac{1}{3}\frac{1}{yx^4}\right) + \frac{5}{21}a^2\frac{1}{y^3x^3} \right], \left[\frac{1}{y^5x^2} - \frac{5}{7}a\frac{1}{y^7x} + \frac{21}{5}a^2\frac{1}{y^2x^3} \right], \left[\frac{1}{y^4x^2} \right], \\ & \left[\frac{1}{y^6x} - \frac{1}{3}a\frac{1}{yx^3} \right], \left[\frac{1}{y^3x^2} \right], \left[\frac{1}{y^5x} \right], \left[\frac{1}{y^2x^2} \right], \left[\frac{1}{y^4x} \right], \left[\frac{1}{yx^2} \right], \left[\frac{1}{y^3x} \right], \left[\frac{1}{y^2x} \right], \left[\frac{1}{yx} \right]. \end{aligned}$$

The first cohomology class generates Σ over $\mathcal{O}_{X,O}$.

The vector space V of differential operators v acting on Σ is generated by the following 14 operators :

$$\begin{aligned} & 252yx\partial_x - 30ax\partial_y + 65a^2y^4\partial_x, 63y^2\partial_y + 15ax\partial_y + 25a^2y^4\partial_x, \\ & 7y^2x\partial_x - 2ayx\partial_y, 7y^3\partial_y + 5ayx\partial_y, 6yx\partial_y - 5ay^5\partial_x, \\ & y^4\partial_y, y^3x\partial_x, y^2x\partial_y, y^5\partial_y, y^4x\partial_x, y^3x\partial_y, y^5x\partial_x, y^4x\partial_y, y^5x\partial_y. \end{aligned}$$

The solution space of the simultaneous homogeneous equation $vh(x, y) = 0$ ($\forall v \in V$) is

$$H = \text{Span}\{1, y^5x\}.$$

Note that $y^5x = f \bmod I$. Since $xf, yf \in I$, we have

$$\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X)) = \text{Span}\{\sigma, \delta\},$$

where δ is the delta function with support at the origin.

4.4 Computations for normal forms

In this section, we give results of computations for normal forms of singularities of modality 1 listed before. To compute the solution space $\text{Hom}_{\mathcal{D}_X}(\mathcal{D}_X/\text{Ann}_{\leq 1}, \mathcal{H}_{[O]}^n(\mathcal{O}_X))$, we give

- partial derivatives f_{z_i} of the function $f(z)$,
- the standard base of the ideal I of partial derivatives of $f(z)$ at the origin,
- basis monomials m_1, \dots, m_μ of \mathcal{O}_X/I and its degree,
- basis $\sigma_1, \dots, \sigma_\mu$ of the vector space Σ and its degree,
- basis v_1, \dots, v_N of the vector space V and its degree,

and H as the solution space of the simultaneous homogeneous equations $vh(z) = 0$ for every $v \in V$. The number below the basis of $\mathcal{O}_{X,O}/I$, Σ , and V is its degree. Here, each z_j has weight α_j and each ∂_j has weight $-\alpha_j$.

We give basis σ_j of Σ which satisfies Proposition 4.1 for every basis monomial $m_{\mu-j+1}$ of $\mathcal{O}_{X,O}/I$, where $\mu = \dim \mathcal{O}_{X,O}/I$ is the Milnor number. That is, $\{\sigma_1, \dots, \sigma_\mu\}$ is the dual base of the monomial base $\{m_\mu, \dots, m_1\}$ of $\mathcal{O}_{X,O}/I$. The cohomology class σ_1 generates Σ over $\mathcal{O}_{X,O}$. Note that in expressions of the basis of Σ , we find the basis of the set of local cohomology classes annihilated by partial derivatives of quasihomogeneous part of the function f if we substitute $a = 0$.

We use the standard basis in computations with respect to the lexicographic order with $z_i \succ z_j$ or $z_i \succ z_j \succ z_k$ of $\alpha_i \geq \alpha_j \geq \alpha_k$ where α_i is the weight of the variable z_i (resp. j, k). Therefore, the monomial basis of the local ring $\mathcal{O}_{X,O}/\langle f_1, \dots, f_n \rangle$ of Z_{12} , Q_{10} , S_{11} and S_{12} used in this paper are different from that in [1].

4.4.1 $E_{12} : x^3 + y^7 + axy^5$

$f = x^3 + y^7 + axy^5$ (of type $(7, 3)$ of degree 21)

Partial derivatives : $f_x = 3x^2 + ay^5$, $f_y = 7y^6 + 5axy^4$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^8, 7y^6 + 5ay^4x, 3x^2 + ay^5\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	y^2	x	y^3	yx	y^4	y^2x	y^5	y^3x	y^4x	y^5x
0	3	6	7	9	10	12	13	15	16	19	22

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{y^6x^2} + a \left(-\frac{5}{7} \frac{1}{y^8x} - \frac{1}{3} \frac{1}{yx^4} \right) + \frac{5}{21} a^2 \frac{1}{y^3x^3} \right], \sigma_2 = \left[\frac{1}{y^5x^2} - \frac{5}{7} a \frac{1}{y^7x} + \frac{21}{5} a^2 \frac{1}{y^2x^3} \right], \\ \sigma_3 &= \left[\frac{1}{y^4x^2} \right], \sigma_4 = \left[\frac{1}{y^6x} - \frac{1}{3} a \frac{1}{yx^3} \right], \sigma_5 = \left[\frac{1}{y^3x^2} \right], \sigma_6 = \left[\frac{1}{y^5x} \right], \sigma_7 = \left[\frac{1}{y^2x^2} \right], \sigma_8 = \left[\frac{1}{y^4x} \right], \\ \sigma_9 &= \left[\frac{1}{yx^2} \right], \sigma_{10} = \left[\frac{1}{y^3x} \right], \sigma_{11} = \left[\frac{1}{y^2x} \right], \sigma_{12} = \left[\frac{1}{yx} \right] \end{aligned}$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
-32	-29	-26	-25	-23	-22	-20	-19	-17	-16	-13	-10

Basis of V :

$$\begin{aligned} v_1 &= 252yx\partial_x - 30ax\partial_y + 65a^2y^4\partial_x, v_2 = 63y^2\partial_y + 15ax\partial_y + 25a^2y^4\partial_x, \\ v_3 &= 7y^2x\partial_x - 2axy\partial_y, v_4 = +7y^3\partial_y + 5axy\partial_y, v_5 = 6yx\partial_y - 5ay^5\partial_x, \\ v_6 &= y^4\partial_y, v_7 = y^3x\partial_x, v_8 = y^2x\partial_y, v_9 = y^5\partial_y, v_{10} = y^4x\partial_x, v_{11} = y^3x\partial_y, \\ v_{12} &= y^5x\partial_x, v_{13} = y^4x\partial_y, v_{14} = y^5x\partial_y \end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}
3	3	6	6	7	9	9	10	12	12	13	15	16	19

Solution space $H : H = \text{Span}\{1, y^5x\}$.

4.4.2 $E_{13} : x^3 + xy^5 + ay^8$

$f = x^3 + xy^5 + ay^8$ (of type (5, 2) of degree 15)

Partial derivatives : $f_x = 3x^2 + y^5$, $f_y = 5xy^4 + 8ay^7$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^9, 5y^4x + 8ay^7, 3x^2 + y^5\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}
1	y	y^2	x	y^3	yx	y^4	y^2x	y^5	y^3x	y^6	y^7	y^8
0	2	4	5	6	7	8	9	10	11	12	14	16

Basis of Σ :

$$\begin{aligned}\sigma_1 &= \left[\frac{1}{y^9x} - \frac{1}{3} \frac{1}{y^4x^3} + a \left(-\frac{8}{5} \frac{1}{y^6x^2} + \frac{8}{15} \frac{1}{yx^4} \right) \right], \sigma_2 = \left[\frac{1}{y^8x} - \frac{1}{3} \frac{1}{y^3x^3} - \frac{8}{5} a \frac{1}{y^5x^2} \right], \\ \sigma_3 &= \left[\frac{1}{y^7x} - \frac{1}{3} \frac{1}{y^2x^3} \right], \sigma_4 = \left[\frac{1}{y^4x^2} \right], \sigma_5 = \left[\frac{1}{y^6x} - \frac{1}{3} \frac{1}{yx^3} \right], \sigma_6 = \left[\frac{1}{y^3x^2} \right], \sigma_7 = \left[\frac{1}{y^5x} \right], \\ \sigma_8 &= \left[\frac{1}{y^2x^2} \right], \sigma_9 = \left[\frac{1}{y^4x} \right], \sigma_{10} = \left[\frac{1}{yx^2} \right], \sigma_{11} = \left[\frac{1}{y^3x} \right], \sigma_{12} = \left[\frac{1}{y^2x} \right], \sigma_{13} = \left[\frac{1}{yx} \right]\end{aligned}$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
-23	-21	-19	-18	-17	-16	-15	-14	-13	-12	-11	-9	-7

Basis of V :

$$\begin{aligned}v_1 &= 125yx\partial_x + 50y^2\partial_y - 40ay^4\partial_x + 192a^2y^2x\partial_x, v_2 = 20y^4\partial_x + 15x\partial_y - 156ay^2x\partial_x, \\ v_3 &= 5y^2x\partial_x + 2y^3\partial_y, v_4 = yx\partial_y - 4ay^3x\partial_x, v_5 = 5y^5\partial_x - 24ay^3x\partial_x, \\ v_6 &= 5y^3x\partial_x + 2y^4\partial_y, v_7 = y^2x\partial_y, v_8 = y^6\partial_x, v_9 = y^5\partial_y, v_{10} = y^3x\partial_y, \\ v_{11} &= y^7\partial_x, v_{12} = y^6\partial_y, v_{13} = y^8\partial_x, v_{14} = y^7\partial_y, v_{15} = y^8\partial_y\end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
2	3	4	5	5	6	7	7	8	9	9	10	11	12	14

Solution space $H : H = \text{Span}\{1, y^8\}$

4.4.3 $E_{14} : x^3 + y^8 + axy^6$

$f = x^3 + y^8 + axy^6$ (of type (8, 3) of degree 24)

Partial derivatives : $f_x = 3x^2 + ay^6$, $f_y = 8y^7 + 6axy^5$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^9, 4y^7 + 3axy^5, 3x^2 + ay^6\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}	m_{14}
1	y	y^2	x	y^3	yx	y^4	y^2x	y^5	y^3x	y^6	y^4x	y^5x	y^6x
0	3	6	8	9	11	12	14	15	17	18	20	23	26

Basis of Σ :

$$\begin{aligned}\sigma_1 &= \left[\frac{1}{y^7x^2} + a \left(-\frac{3}{4} \frac{1}{y^9x} - \frac{1}{3} \frac{1}{yx^4} \right) + \frac{1}{4} a^2 \frac{1}{y^3x^3} \right], \sigma_2 = \left[\frac{1}{y^6x^2} - \frac{3}{4} a \frac{1}{y^8x} + \frac{1}{4} a^2 \frac{1}{y^2x^3} \right], \\ \sigma_3 &= \left[\frac{1}{y^5x^2} \right], \sigma_4 = \left[\frac{1}{y^7x} - \frac{1}{3} a \frac{1}{yx^3} \right], \sigma_5 = \left[\frac{1}{y^4x^2} \right], \sigma_6 = \left[\frac{1}{y^6x} \right], \sigma_7 = \left[\frac{1}{y^3x^2} \right], \sigma_8 = \left[\frac{1}{y^5x} \right], \\ \sigma_9 &= \left[\frac{1}{y^2x^2} \right], \sigma_{10} = \left[\frac{1}{y^4x} \right], \sigma_{11} = \left[\frac{1}{yx^2} \right], \sigma_{12} = \left[\frac{1}{y^3x} \right], \sigma_{13} = \left[\frac{1}{y^2x} \right], \sigma_{14} = \left[\frac{1}{yx} \right]\end{aligned}$$

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}	σ_{14}
-37	-34	-31	-29	-28	-26	-25	-23	-22	-20	-19	-17	-14	-11

Basis of V :

$$\begin{aligned} v_1 &= 28yx\partial_x - 3ax\partial_y - 4a^2y^5\partial_x, v_2 = 28y^2\partial_y + 6ax\partial_y + 15a^2y^5\partial_x, \\ v_3 &= 4y^2x\partial_x - ayx\partial_y, v_4 = 4y^3\partial_y + 3ayx\partial_y, v_5 = yx\partial_y - ay^6\partial_x, \\ v_6 &= y^4\partial_y, v_7 = y^3x\partial_x, v_8 = y^2x\partial_y, v_9 = y^5\partial_y, v_{10} = y^4x\partial_x, v_{11} = y^3x\partial_y, \\ v_{12} &= y^6\partial_y, v_{13} = y^5x\partial_x, v_{14} = y^4x\partial_y, v_{15} = y^6x\partial_x, v_{16} = y^5x\partial_y, v_{17} = y^6x\partial_y \end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
3	3	6	6	8	9	9	11	12	12	14	15	15	17	18	20	23

Solution space H : $H = \text{Span}\{1, y^6x\}$

4.4.4 $Z_{11} : x^3y + y^5 + axy^4$

$f = x^3y + y^5 + axy^4$ (of type (4, 3) of degree 15)

Partial derivatives : $f_x = 3x^2y + ay^4$, $f_y = x^3 + 5y^4 + 4axy^3$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^6, 15y^5 + 11ay^4x, 3yx^2 + ay^4, x^3 + 5y^4 + 4axy^3\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
1	y	x	y^2	yx	x^2	y^3	y^2x	y^4	y^3x	y^4x
0	3	4	6	7	8	9	10	12	13	16

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{y^5x^2} - 5\frac{1}{yx^5} + a\left(-\frac{11}{15}\frac{1}{y^6x} - \frac{1}{3}\frac{1}{y^2x^4}\right) + \frac{11}{45}a^2\frac{1}{y^3x^3} \right], \sigma_2 = \left[\frac{1}{y^4x^2} - \frac{4}{5}a\frac{1}{y^5x} + \frac{4}{15}a^2\frac{1}{y^2x^3} \right], \\ \sigma_3 &= \left[\frac{1}{y^5x} - 5\frac{1}{yx^4} - \frac{1}{3}a\frac{1}{y^2x^3} \right], \sigma_4 = \left[\frac{1}{y^3x^2} \right], \sigma_5 = \left[\frac{1}{y^4x} \right], \sigma_6 = \left[\frac{1}{yx^3} \right], \sigma_7 = \left[\frac{1}{y^2x^2} \right], \\ \sigma_8 &= \left[\frac{1}{y^3x} \right], \sigma_9 = \left[\frac{1}{yx^2} \right], \sigma_{10} = \left[\frac{1}{y^2x} \right], \sigma_{11} = \left[\frac{1}{yx} \right] \\ \sigma_1 & \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \quad \sigma_6 \quad \sigma_7 \quad \sigma_8 \quad \sigma_9 \quad \sigma_{10} \quad \sigma_{11} \\ -23 & \quad -20 \quad -19 \quad -17 \quad -16 \quad -15 \quad -14 \quad -13 \quad -11 \quad -10 \quad -7 \end{aligned}$$

Basis of V :

$$\begin{aligned} v_1 &= 15yx\partial_x - a(61x^2\partial_x + 48yx\partial_y), v_2 = 15y^2\partial_y + a(108x^2\partial_x + 83yx\partial_y), \\ v_3 &= 60x^2\partial_x + 45yx\partial_y + ax^2\partial_y, v_4 = 5y^3\partial_x + 2x^2\partial_y, \\ v_5 &= y^3\partial_y, v_6 = y^2x\partial_x, v_7 = y^2x\partial_y, v_8 = y^4\partial_x, v_9 = y^4\partial_y, \\ v_{10} &= y^3x\partial_x, v_{11} = y^3x\partial_y, v_{12} = y^4x\partial_x, v_{13} = y^4x\partial_y \end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}
3	3	4	5	6	6	7	8	9	9	10	12	13

Solution space H : $H = \text{Span}\{1, y^4x\}$

4.4.5 $Z_{12} : x^3y + xy^4 + ay^6$

$f = x^3y + xy^4 + ay^6$ (of type (3, 2) of degree 11)

Partial derivatives : $f_x = 3x^2y + y^4$, $f_y = x^3 + 4xy^3 + 6ay^5$

The standard base of $I = \langle f_x, f_y \rangle_O$:

$$\{y^7, 33y^4x - 7ay^6, 3yx^2 + y^4 + 2ay^3x, 33x^3 + 132y^3x - 33ay^5 - 14a^3y^6\}$$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	x	y^2	yx	x^2	y^3	y^2x	y^4	y^3x	y^5	y^6
0	2	3	4	5	6	6	7	8	9	10	12

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{y^7x} - \frac{1}{3} \frac{1}{y^4x^3} + \frac{4}{3} \frac{1}{yx^5} + a \left(\frac{6}{11} \frac{1}{y^2x^4} - \frac{18}{11} \frac{1}{y^5x^2} \right) \right], \sigma_2 = \left[\frac{1}{y^6x} - \frac{1}{3} \frac{1}{y^3x^3} - \frac{3}{2} a \frac{1}{y^4x^2} \right], \\ \sigma_3 &= \left[\frac{1}{y^4x^2} - 4 \frac{1}{yx^4} \right], \sigma_4 = \left[\frac{1}{y^5x} - \frac{1}{3} \frac{1}{y^2x^3} \right], \sigma_5 = \left[\frac{1}{y^3x^2} \right], \sigma_6 = \left[\frac{1}{y^4x} \right], \sigma_7 = \left[\frac{1}{yx^3} \right], \\ \sigma_8 &= \left[\frac{1}{y^2x^2} \right], \sigma_9 = \left[\frac{1}{y^3x} \right], \sigma_{10} = \left[\frac{1}{yx^2} \right], \sigma_{11} = \left[\frac{1}{y^2x} \right], \sigma_{12} = \left[\frac{1}{yx} \right] \end{aligned}$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
-17	-15	-14	-13	-12	-11	-11	-10	-9	-8	-7	-5

Basis of V :

$$\begin{aligned} v_1 &= 1936y^2\partial_x - 13068axy\partial_x + 2673a^2x^2\partial_x + 4374a^3y^2x\partial_x + 1452x\partial_y, \\ v_2 &= 132yx\partial_x + 88y^2\partial_y - 99ax^2\partial_x - 162a^2y^2x\partial_x, v_3 = 3x^2\partial_x + 2yx\partial_y, \\ v_4 &= 4y^3\partial_x + 3yx\partial_y - 27ay^2x\partial_x, v_5 = 3y^2x\partial_x + 2y^3\partial_y, v_6 = 8y^3\partial_y - 9x^2\partial_y, \\ v_7 &= y^2x\partial_y, v_8 = y^4\partial_x, v_9 = y^4\partial_y, v_{10} = y^3x\partial_x, v_{11} = y^3x\partial_y, \\ v_{12} &= y^5\partial_x, v_{13} = y^5\partial_y, v_{14} = y^6\partial_x, v_{15} = y^6\partial_y \end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}
1	2	3	3	4	4	5	5	6	6	7	7	8	9	10

Solution space $H : H = \text{Span}\{1, y^6\}$

4.4.6 $Z_{13} : x^3y + y^6 + axy^5$

$f = x^3y + y^6 + axy^5$ (of type (5, 3) of degree 18)

Partial derivatives : $f_x = 3x^2y + ay^5$, $f_y = x^3 + 6y^5 + 5axy^4$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^7, 9y^6 + 7ay^5x, 3yx^2 + ay^5, x^3 + 6y^5 + 5axy^4x\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}
1	y	x	y^2	yx	y^3	x^2	y^2x	y^4	y^3x	y^5	y^4x	y^5x
0	3	5	6	8	9	10	11	12	14	15	17	20

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{y^6x^2} - 6 \frac{1}{yx^5} + a \left(-\frac{7}{9} \frac{1}{y^7x} + \frac{7}{27} \frac{1}{y^3x^3} - \frac{1}{3} \frac{1}{y^2x^4} \right) \right], \sigma_2 = \left[\frac{1}{y^5x^2} - \frac{5}{6} a \frac{1}{y^6x} + \frac{5}{18} a^2 \frac{1}{y^2x^3} \right] \\ \sigma_3 &= \left[\frac{1}{y^5x^2} - 5a \frac{1}{yx^4} \right], \sigma_4 = \left[\frac{1}{y^4x^2} \right], \sigma_5 = \left[\frac{1}{y^5x} \right], \sigma_6 = \left[\frac{1}{y^3x^2} \right], \sigma_7 = \left[\frac{1}{yx^3} \right], \sigma_8 = \left[\frac{1}{y^4x} \right] \end{aligned}$$

$$\sigma_9 = \left[\frac{1}{y^2 x^2} \right], \sigma_{10} = \left[\frac{1}{y^3 x} \right], \sigma_{11} = \left[\frac{1}{y x^2} \right], \sigma_{12} = \left[\frac{1}{y^2 x} \right], \sigma_{13} = \left[\frac{1}{y x} \right]$$

Degrees :

$$\begin{array}{cccccccccccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} & \sigma_{13} \\ -28 & -25 & -23 & -22 & -20 & -19 & -18 & -17 & -16 & -14 & -13 & -11 & -8 \end{array}$$

Basis of V :

$$\begin{aligned} v_1 &= 9yx\partial_x - a(23x^2\partial_x + 15yx\partial_y), v_2 = 9y^2\partial_y + a(60x^2\partial_x + 37yx\partial_y), \\ v_3 &= 45x^2\partial_x + 27yx\partial_y + ax^2\partial_y, v_4 = y^3\partial_y, v_5 = y^2x\partial_x, v_6 = 3y^4\partial_x + x^2\partial_y, \\ v_7 &= y^2x\partial_y, v_8 = y^4\partial_y, v_9 = y^3x\partial_x, v_{10} = y^5\partial_x, v_{11} = y^3x\partial_y, v_{12} = y^5\partial_y, \\ v_{13} &= y^4x\partial_x, v_{14} = y^4x\partial_y, v_{15} = y^5x\partial_x, v_{16} = y^5x\partial_y \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} \\ 3 & 3 & 5 & 6 & 6 & 7 & 8 & 9 & 9 & 10 & 11 & 12 & 12 & 14 & 15 & 17 \end{array}$$

Solution space $H : H = \text{Span}\{1, y^5 x\}$

4.4.7 $W_{12} : x^4 + y^5 + ax^2 y^3$

$f = x^4 + y^5 + ax^2 y^3$ (of type (5, 4) of degree 20)

Partial derivatives : $f_x = 4x^3 + 2axy^3, f_y = 5y^4 + 3ax^2 y^2$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^6, y^4 x, 5y^4 + 3ay^2 x^2, 2x^3 + ay^3 x\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

$$\begin{array}{cccccccccccccc} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & m_{10} & m_{11} & m_{12} \\ 1 & y & x & y^2 & yx & x^2 & y^3 & y^2 x & yx^2 & y^3 x & y^2 x^2 & y^3 x^2 \\ 0 & 4 & 5 & 8 & 9 & 10 & 12 & 13 & 14 & 17 & 18 & 22 \end{array}$$

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{y^4 x^3} + a \left(-\frac{3}{5} \frac{1}{y^6 x} - \frac{1}{2} \frac{1}{y x^5} \right) \right], \sigma_2 = \left[\frac{1}{y^3 x^3} - \frac{3}{5} a \frac{1}{y^5 x} \right], \sigma_3 = \left[\frac{1}{y^4 x^2} - \frac{1}{2} a \frac{1}{y x^4} \right], \\ \sigma_4 &= \left[\frac{1}{y^2 x^3} \right], \sigma_5 = \left[\frac{1}{y^3 x^2} \right], \sigma_6 = \left[\frac{1}{y^4 x} \right], \sigma_7 = \left[\frac{1}{y x^3} \right], \sigma_8 = \left[\frac{1}{y^2 x^2} \right], \sigma_9 = \left[\frac{1}{y^3 x} \right], \\ \sigma_{10} &= \left[\frac{1}{y x^2} \right], \sigma_{11} = \left[\frac{1}{y^2 x} \right], \sigma_{12} = \left[\frac{1}{y x} \right] \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} \\ -31 & -27 & -26 & -23 & -22 & -21 & -19 & -18 & -17 & -14 & -13 & -9 \end{array}$$

Basis of V :

$$\begin{aligned} v_1 &= 10yx\partial_x - 3ax^2\partial_y, v_2 = 10y^2\partial_y + 3ax^2\partial_y, v_3 = 3x^2\partial_x + ay^3\partial_x, v_4 = 2yx\partial_y - ay^3\partial_x, \\ v_5 &= y^3\partial_y, v_6 = y^2x\partial_x, v_7 = y^2x\partial_y, v_8 = yx^2\partial_x, v_9 = yx^2\partial_y, v_{10} = y^3x\partial_x, \\ v_{11} &= y^3x\partial_y, v_{12} = y^2x^2\partial_x, v_{13} = y^2x^2\partial_y, v_{14} = y^3x^2\partial_x, v_{15} = y^3x^2\partial_y \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} \\ 4 & 4 & 5 & 5 & 8 & 8 & 9 & 9 & 10 & 12 & 13 & 13 & 14 & 17 & 18 \end{array}$$

Solution space $H : H = \text{Span}\{1, y^3 x^2\}$

4.4.8 $W_{13} : x^4 + xy^4 + ay^6$

$f = x^4 + xy^4 + ay^6$ (of type (4, 3) of degree 16)

Partial derivatives : $f_x = 4x^3 + y^4$, $f_y = 4xy^3 + 6ay^5$

The standard base of $I = \langle f_x, f_y \rangle_O : \{y^7, 2y^3x + 3ay^5, 4x^3 + y^4\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}	m_{13}
1	y	x	y^2	yx	x^2	y^3	y^2x	yx^2	y^4	y^2x^2	y^5	y^6
0	3	4	6	7	8	9	10	11	12	14	15	18

Basis of Σ :

$$\begin{aligned}\sigma_1 &= \left[\frac{1}{y^7x} - \frac{1}{4} \frac{1}{y^3x^4} + a \left(-\frac{3}{2} \frac{1}{y^5x^2} + \frac{3}{8} \frac{1}{y^5x^5} \right) \right], \sigma_2 = \left[\frac{1}{y^6x} - \frac{1}{4} \frac{1}{y^2x^4} - \frac{3}{2} a \frac{1}{y^4x^2} \right], \sigma_3 = \left[\frac{1}{y^3x^3} \right], \\ \sigma_4 &= \left[\frac{1}{y^5x} - \frac{1}{4} \frac{1}{yx^4} \right], \sigma_5 = \left[\frac{1}{y^2x^3} \right], \sigma_6 = \left[\frac{1}{y^3x^2} \right], \sigma_7 = \left[\frac{1}{y^4x} \right], \sigma_8 = \left[\frac{1}{yx^3} \right], \\ \sigma_9 &= \left[\frac{1}{y^2x^2} \right], \sigma_{10} = \left[\frac{1}{y^3x} \right], \sigma_{11} = \left[\frac{1}{yx^2} \right], \sigma_{12} = \left[\frac{1}{y^2x} \right], \sigma_{13} = \left[\frac{1}{yx} \right]\end{aligned}$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}	σ_{13}
-25	-22	-21	-19	-18	-17	-16	-15	-14	-13	-11	-10	-7

Basis of V :

$$\begin{aligned}v_1 &= 4yx\partial_x + 3y^2\partial_y - 3ay^3\partial_x, v_2 = 8x^2\partial_x - 9ay^3\partial_y, v_3 = 2yx\partial_y + 3ay^3\partial_y, \\ v_4 &= 3y^3\partial_x + 4x^2\partial_y, v_5 = 3y^3\partial_y + 4y^2x\partial_x, v_6 = y^2x\partial_y, v_7 = yx^2\partial_x, \\ v_8 &= yx^2\partial_y, v_9 = y^4\partial_x, v_{10} = y^4\partial_y, v_{11} = y^2x^2\partial_x, v_{12} = y^2x^2\partial_y, \\ v_{13} &= y^5\partial_x, v_{14} = y^5\partial_y, v_{15} = y^6\partial_x, v_{16} = y^6\partial_y\end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
3	4	4	5	6	7	7	8	8	9	10	11	11	12	14	15

Solution space $H : H = \text{Span}\{1, y^6\}$

4.4.9 $Q_{10} : x^3 + y^4 + yz^2 + axy^3$

$f = x^3 + y^4 + yz^2 + axy^3$ (of type (8, 6, 9) of degree 24)

Partial derivatives : $f_x = 3x^2 + ay^3$, $f_y = 4y^3 + z^2 + 3axy^2$, $f_z = 2yz$

The standard base of $I = \langle f_x, f_y \rangle_O : \{x^4, 4yx^2 + 3ax^3, 3x^2 + ay^3, zx^2, zy, 12x^2 - az^2 - 3a^2y^2x\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}
1	y	x	z	y^2	yx	zx	y^3	y^2x	y^3x
0	6	8	9	12	14	17	18	20	26

Basis of Σ :

$$\begin{aligned}\sigma_1 &= \left[\frac{1}{zy^4x^2} - 4 \frac{1}{z^3yx^2} + a \left(-\frac{3}{4} \frac{1}{zy^5x} - \frac{1}{3} \frac{1}{zy^4x^4} \right) + \frac{1}{4} a^2 \frac{1}{zy^2x^3} \right], \sigma_2 = \left[\frac{1}{zy^3x^2} - \frac{3}{4} a \frac{1}{zy^4x} - \frac{1}{4} a^2 \frac{1}{zyx} \right], \\ \sigma_3 &= \left[\frac{1}{zy^4x} - 4 \frac{1}{z^3yx} + \frac{1}{3} a \frac{1}{zyx^3} \right], \sigma_4 = \left[\frac{1}{z^2yx^2} \right], \sigma_5 = \left[\frac{1}{zy^2x^2} \right], \sigma_6 = \left[\frac{1}{zy^3x} \right],\end{aligned}$$

$$\sigma_7 = \left[\frac{1}{z^2 y x} \right], \sigma_8 = \left[\frac{1}{z y x^2} \right], \sigma_9 = \left[\frac{1}{z y^2 x} \right], \sigma_{10} = \left[\frac{1}{z y x} \right]$$

Degrees :

$$\begin{array}{cccccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} \\ -49 & -43 & -41 & -40 & -37 & -35 & -32 & -31 & -29 & -23 \end{array}$$

Basis of V :

$$\begin{aligned} v_1 &= z\partial_y + 4y^2\partial_z + 3axy\partial_x, v_2 = 8yx\partial_x - a(4yx\partial_y + 3zx\partial_z), \\ v_3 &= 4y^2\partial_y + a(3yx\partial_y + 3zx\partial_z), v_4 = 4yx\partial_y + 6zx\partial_z - 2ay^3\partial_x, \\ v_5 &= zx\partial_x, v_6 = 4y^3\partial_z + 3ay^2x\partial_z, v_7 = zx\partial_y + 4y^2x\partial_z, \\ v_8 &= y^3\partial_y, v_9 = y^2x\partial_x, v_{10} = y^2x\partial_y, v_{11} = y^3x\partial_z, v_{12} = y^3x\partial_x, v_{13} = y^3x\partial_y \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} \\ 3 & 6 & 6 & 8 & 9 & 9 & 11 & 12 & 12 & 14 & 17 & 18 & 20 \end{array}$$

Solution space $H : H = \text{Span}\{1, y^3x\}$

4.4.10 $Q_{11} : x^3 + y^2z + xz^3 + az^5$

$f = x^3 + y^2z + xz^3 + az^5$ (of type (6, 7, 4) of degree 18)

Partial derivatives : $f_x = 3x^2 + z^3, f_y = 2yz, f_z = y^2 + 3xz^2 + 5az^4$

The standard base of $I = \langle f_x, f_y \rangle_O : \{x^4, yx^2, 9x^3 - 5ay^2x, y^3, zx^3, zy, 3z^2x + y^2 - 15azx^2, 3x^2 + z^3\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

$$\begin{array}{cccccccccccc} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & m_{10} & m_{11} \\ 1 & z & x & y & z^2 & zx & z^3 & yx & z^2x & z^4 & z^5 \\ 0 & 4 & 6 & 7 & 8 & 10 & 12 & 13 & 4 & 16 & 20 \end{array}$$

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{z^6 y x} + \frac{1}{z y^3 x^2} - \frac{1}{3} \frac{1}{z^3 y x^3} + a \left(-\frac{5}{3} \frac{1}{z^4 y x^2} + \frac{5}{9} \frac{1}{z y x^4} \right) \right], \sigma_2 = \left[\frac{1}{z^5 y x} - \frac{1}{3} \frac{1}{z^2 y x^3} - \frac{5}{3} a \frac{1}{z^3 y x^2} \right], \\ \sigma_3 &= \left[\frac{1}{z y^3 x} - \frac{1}{3} \frac{1}{z^3 y x^2} \right], \sigma_4 = \left[\frac{1}{z y^2 x^2} \right], \sigma_5 = \left[\frac{1}{z^4 y x} - \frac{1}{3} \frac{1}{z y x^3} \right], \sigma_6 = \left[\frac{1}{z^2 y x^2} \right], \\ \sigma_7 &= \left[\frac{1}{z^3 y x} \right], \sigma_8 = \left[\frac{1}{z y^2 x} \right], \sigma_9 = \left[\frac{1}{z y x^2} \right], \sigma_{10} = \left[\frac{1}{z^2 y x} \right], \sigma_{11} = \left[\frac{1}{z y x} \right] \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} \\ -37 & -33 & -31 & -30 & -29 & -27 & -25 & -24 & -23 & -21 & -17 \end{array}$$

Basis of V :

$$\begin{aligned} v_1 &= 3zx\partial_y + y\partial_z + 5az^3\partial_y, v_2 = 6zx\partial_x + 4z^2\partial_z - 5ayx\partial_y, \\ v_3 &= 2yx\partial_y + 2zx\partial_z - 5az^2x\partial_x, v_4 = 6zx\partial_z + 2z^3\partial_x - a(15z^2x\partial_x + 10z^2x\partial_z), \\ v_5 &= yx\partial_x, v_6 = 3z^2x\partial_y + 5az^4\partial_y, v_7 = 3z^2x\partial_x + 2z^3\partial_z, v_8 = z^4\partial_y - yx\partial_z, \\ v_9 &= z^2x\partial_z, v_{10} = z^4\partial_x, v_{11} = z^4\partial_z, v_{12} = z^5\partial_y, v_{13} = z^5\partial_x, v_{14} = z^5\partial_z \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} \\ 3 & 4 & 6 & 6 & 7 & 7 & 8 & 9 & 10 & 10 & 12 & 13 & 14 & 16 \end{array}$$

Solution space $H : H = \text{Span}\{1, z^5\}$

4.4.11 $Q_{12} : x^3 + y^5 + yz^2 + axy^4$

$f = x^3 + y^5 + yz^2 + axy^4$ (of type (5, 3, 6) of degree 15)

Partial derivatives : $f_x = 3x^2 + ay^4$, $f_y = 5y^4 + z^2 + 4axy^3$, $f_z = 2yz$

The standard base of $I = \langle f_x, f_y \rangle_O : \{x^4, 5yx^2 + 4ax^3, 3x^2 + ay^4, zx^2, zy, 15x^2 - az^2 - 4a^2y^3x\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	y	x	z	y ²	yx	y ³	zx	y ² x	y ⁴	y ³ x	y ⁴ x
0	3	5	6	6	8	9	11	11	12	14	17

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{zy^5x^2} - 5\frac{1}{z^3yx^2} + a\left(-\frac{4}{5}\frac{1}{zy^6x} - \frac{1}{3}\frac{1}{zyx^4}\right) + \frac{4}{15}a^2\frac{1}{zy^2x^3} \right], \sigma_2 = \left[\frac{1}{zy^4x^2} - \frac{4}{5}a\frac{1}{zy^5x} + \frac{4}{15}a^2\frac{1}{zyx^3} \right],$$

$$\sigma_3 = \left[\frac{1}{zy^5x} - 5\frac{1}{z^3yx} - \frac{a}{3}\frac{1}{zyx^3} \right], \sigma_4 = \left[\frac{1}{zy^3x^2} \right], \sigma_5 = \left[\frac{1}{z^2yx^2} \right], \sigma_6 = \left[\frac{1}{zy^4x} \right], \sigma_7 = \left[\frac{1}{zy^2x^2} \right],$$

$$\sigma_8 = \left[\frac{1}{zy^3x} \right], \sigma_9 = \left[\frac{1}{z^2yx} \right], \sigma_{10} = \left[\frac{1}{zyx^2} \right], \sigma_{11} = \left[\frac{1}{zy^2x} \right], \sigma_{12} = \left[\frac{1}{zyx} \right],$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
-31	-28	-26	-25	-25	-23	-22	-20	-20	-19	-17	-14

Basis of V :

$$v_1 = 5yx\partial_x - a(2yx\partial_y + 2zx\partial_z), v_2 = z\partial_y + 5y^3\partial_z + 4ay^2x\partial_x,$$

$$v_3 = 5y^2\partial_y + a(4yx\partial_y + 6zx\partial_z), v_4 = 3yx\partial_y + 6zx\partial_z - 2ay^4\partial_x,$$

$$v_5 = y^3\partial_y, v_6 = y^2x\partial_x, v_7 = zx\partial_x, v_8 = 5y^4\partial_z + 4ay^3x\partial_x,$$

$$v_9 = y^2x\partial_y, v_{10} = zx\partial_y + 5y^3x\partial_z, v_{11} = y^4\partial_y, v_{12} = y^3x\partial_x,$$

$$v_{13} = y^3x\partial_y, v_{14} = y^4x\partial_z, v_{15} = y^4x\partial_x, v_{16} = y^4x\partial_y,$$

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}
3	3	3	5	6	6	6	6	8	8	9	9	11	11	12	14

Solution space $H : H = \text{Span}\{1, y^4x\}$

4.4.12 $S_{11} : x^4 + y^2z + xz^2 + ay^2x^2$

$f = x^4 + y^2z + xz^2 + ay^2x^2$ (of type (4, 5, 6) of degree 16)

Partial derivatives : $f_x = 4x^3 + z^2 + 2ay^2x$, $f_y = 2zy + 2ayx^2$, $f_z = y^2 + 2zx$

The standard base of $I = \langle f_x, f_y \rangle_O :$

$$\{4x^5 + 5a^2x^6, 4yx^3 + 5a^2yx^4, 8x^4 + 5ay^2x^2, y^3 - 2ayx^3, 2zx + y^2, zy + ayx^2, 4x^3 + z^2 + 2ay^2x\}$$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
1	x	y	z	x ²	yx	y ²	x ³	yx ²	y ² x	y ² x ²
0	4	5	6	8	9	10	12	13	14	18

Basis of Σ :

$$\sigma_1 = \left[\frac{1}{zy^3x^3} + 2\frac{1}{z^4yx} - \frac{1}{2}\frac{1}{z^2yx^4} + a\left(-\frac{1}{z^2y^3x} + \frac{1}{2}\frac{1}{z^3yx^2} - \frac{5}{8}\frac{1}{zyx^5}\right) \right],$$

$$\sigma_2 = \left[\frac{1}{zy^3x^2} - \frac{1}{2}\frac{1}{z^2yx^3} - \frac{1}{2}a\frac{1}{zyx^4} \right], \sigma_3 = \left[\frac{1}{zy^2x^3} - a\frac{1}{z^2y^2x} \right], \sigma_4 = \left[\frac{1}{z^3yx} - \frac{1}{4}\frac{1}{zyx^4} \right],$$

$$f\sigma_5 = \left[-\frac{1}{zy^3x} + \frac{1}{2}\frac{1}{z^2yx^2} \right], \sigma_6 = \left[\frac{1}{zy^2x^2} \right], \sigma_7 = \left[\frac{1}{zyx^3} \right], \sigma_8 = \left[\frac{1}{z^2yx} \right],$$

$$\sigma_9 = \left[\frac{1}{zy^2x} \right], \sigma_{10} = \left[\frac{1}{zyx^2} \right], \sigma_{11} = \left[\frac{1}{zyx} \right]$$

Degrees :

$$\begin{array}{cccccccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} \\ -33 & -29 & -28 & -27 & -25 & -24 & -23 & -21 & -20 & -19 & -15 \end{array}$$

Basis of V :

$$\begin{aligned} v_1 &= 64x^2\partial_x + 32y^2\partial_z + 144ax^3\partial_z - 65a^2y^2x\partial_z, \\ v_2 &= 16yx\partial_y - 16y^2\partial_z - 48ax^3\partial_z + 15a^2y^2x\partial_z, v_3 = y^2\partial_y, \\ v_4 &= yx\partial_x, v_5 = 4y^2\partial_x + 48x^3\partial_z - 21ay^2x\partial_z, v_6 = 2x^3\partial_y + yx^2\partial_z, \\ v_7 &= 2x^3\partial_x + y^2x\partial_z, v_8 = yx^2\partial_y - y^2x\partial_z, v_9 = yx^2\partial_x, v_{10} = y^2x\partial_y, \\ v_{11} &= y^2x\partial_x, v_{12} = y^2x^2\partial_z, v_{13} = y^2x^2\partial_y, v_{14} = y^2x^2\partial_x \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} \\ 4 & 4 & 5 & 5 & 6 & 7 & 8 & 8 & 9 & 9 & 10 & 12 & 13 & 14 \end{array}$$

Solution space H : $H = \text{Span}\{1, y^2x^2\}$

4.4.13 S_{12} : $x^2y + y^2z + xz^3 + az^5$

$f = x^2y + y^2z + xz^3 + az^5$ (of type (4, 5, 3) of degree 13)

Partial derivatives : $f_x = 2xy + z^3$, $f_y = x^2 + 2yz$, $f_z = y^2 + 3xz^2 + 5az^4$

The standard base of $I = \langle f_x, f_y \rangle_O$:

$$\{x^4, yx^3, y^2x, 39yx^2 - 20ay^3, 13yx^2 - 10azx^3, x^2 + 2zy, 3z^2x + y^2 + 5ax^3, 2yx + z^3\}$$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

$$\begin{array}{cccccccccccccc} m_1 & m_2 & m_3 & m_4 & m_5 & m_6 & m_7 & m_8 & m_9 & m_{10} & m_{11} & m_{12} \\ 1 & z & x & y & z^2 & zx & x^2 & z^3 & z^2x & zx^2 & z^4 & z^5 \\ 0 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 15 \end{array}$$

Basis of Σ :

$$\begin{aligned} \sigma_1 &= \left[\frac{1}{z^6yx} + \frac{3}{2} \frac{1}{zy^4x} - \frac{1}{2} \frac{1}{z^3y^2x^2} + \frac{1}{z^2yx^4} + a \left(-\frac{5}{13} \frac{1}{z^2y^3x} - \frac{20}{13} \frac{1}{z^4yx^2} + \frac{10}{13} \frac{1}{zy^2x^3} \right) \right], \\ \sigma_2 &= \left[\frac{1}{z^5yx} - \frac{1}{2} \frac{1}{z^2y^2x^2} + \frac{1}{zyx^4} - \frac{5}{3} a \frac{1}{z^3yx^2} \right], \sigma_3 = \left[\frac{1}{z^2yx^3} - \frac{1}{2} \frac{1}{z^3y^2x} \right], \sigma_4 = \left[\frac{1}{z^3yx^2} - 3 \frac{1}{zy^3x} \right], \\ \sigma_5 &= \left[\frac{1}{z^4yx} - \frac{1}{2} \frac{1}{zy^2x^2} \right], \sigma_6 = \left[\frac{1}{zyx^3} - \frac{1}{2} \frac{1}{z^2y^2x} \right], \sigma_7 = \left[\frac{1}{z^2yx^2} \right], \sigma_8 = \left[\frac{1}{z^3yx} \right], \\ \sigma_9 &= \left[\frac{1}{zy^2x} \right], \sigma_{10} = \left[\frac{1}{zyx^2} \right], \sigma_{11} = \left[\frac{1}{z^2yx} \right], \sigma_{12} = \left[\frac{1}{zyx} \right] \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccc} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 & \sigma_{10} & \sigma_{11} & \sigma_{12} \\ -27 & -24 & -23 & -22 & -21 & -20 & -19 & -18 & -17 & -16 & -15 & -12 \end{array}$$

Basis of V :

$$\begin{aligned} v_1 &= 39z^2\partial_x + 39zx\partial_y + 26y\partial_z + 130az^3\partial_y + 200a^2zx^2\partial_y, \\ v_2 &= 104zx\partial_x - 65x^2\partial_y + 78z^2\partial_z + 260az^2x\partial_y + 400a^2z^4\partial_y, \\ v_3 &= 13z^3\partial_y + 26zx\partial_z + 80azx^2\partial_y, v_4 = 13x^2\partial_x - 13z^3\partial_y - 30azx^2\partial_y, \\ v_5 &= 78z^2x\partial_y + 13x^2\partial_z + 110az^4\partial_y, v_6 = 26z^3\partial_x - 13x^2\partial_z + 50az^4\partial_y, \\ v_7 &= 3zx^2\partial_y - 2z^3\partial_z, v_8 = 2z^2x\partial_x + zx^2\partial_y, v_9 = z^4\partial_y + 2z^2x\partial_z, v_{10} = zx^2\partial_x - z^4\partial_y, \\ v_{11} &= zx^2\partial_z, v_{12} = z^4\partial_x, v_{13} = z^4\partial_z, v_{14} = z^5\partial_y, v_{15} = z^5\partial_x, v_{16} = z^5\partial_z \end{aligned}$$

Degrees :

$$\begin{array}{cccccccccccccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 & v_7 & v_8 & v_9 & v_{10} & v_{11} & v_{12} & v_{13} & v_{14} & v_{15} & v_{16} \\ 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 8 & 8 & 8 & 10 & 11 & 12 \end{array}$$

Solution space H : $H = \text{Span}\{1, z^5\}$

4.4.14 $U_{12} : x^3 + y^3 + z^4 + axyz^2$

$f = x^3 + y^3 + z^4 + axyz^2$ (of type (4, 4, 3) of degree 12)

Partial derivatives : $f_x = 3x^2 + ayz^2$, $f_y = 3y^2 + axz^2$, $f_z = 4z^3 + 2axyz$

The standard base of $I = \langle f_x, f_y \rangle_O : \{x^4, yx^2, y^2x, x^3 - y^3, zx^2, zy^2, 3y^2 + az^2x, 3x^2 + az^2y, 2z^3 + azyx\}$

Basis of the local ring $\mathcal{O}_{X,O}/I$ and its degrees :

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
1	z	y	x	z^2	zy	zx	yx	z^2y	z^2x	zyx	z^2yx
0	3	4	4	6	7	7	8	10	10	11	14

Basis of Σ :

$$\begin{aligned}\sigma_1 &= \left[\frac{1}{z^3y^2x^2} - \frac{1}{3}a \frac{1}{zyx^4} + a \left(-\frac{1}{2} \frac{1}{z^5yx} - \frac{1}{3} \frac{1}{zy^4x} \right) \right], \sigma_2 = \left[\frac{1}{z^2y^2x^2} - \frac{1}{2}a \frac{1}{z^4yx} \right], \\ \sigma_3 &= \left[\frac{1}{z^3y^2x} - \frac{1}{3}a \frac{1}{zyx^3} \right], \sigma_4 = \left[\frac{1}{z^3yx^2} - \frac{1}{3}a \frac{1}{zy^3x} \right], \sigma_5 = \left[\frac{1}{zy^2x^2} \right], \sigma_6 = \left[\frac{1}{z^2yx^2} \right], \\ \sigma_7 &= \left[\frac{1}{z^2y^2x} \right], \sigma_8 = \left[\frac{1}{z^3yx} \right], \sigma_9 = \left[\frac{1}{zyx^2} \right], \sigma_{10} = \left[\frac{1}{zy^2x} \right], \sigma_{11} = \left[\frac{1}{z^2yx} \right], \sigma_{12} = \left[\frac{1}{zyx} \right]\end{aligned}$$

Degrees :

σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}	σ_{12}
-25	-22	-21	-21	-19	-18	-18	-17	-15	-15	-14	-11

Basis of V :

$$\begin{aligned}v_1 &= 6zx\partial_x - ayz\partial_z, v_2 = 6zy\partial_y - ayz\partial_z, v_3 = -3z^2\partial_z - ayz\partial_z, v_4 = 6yx\partial_x - az^2x\partial_y, \\ v_5 &= -3zx\partial_x + az^2y\partial_z, v_6 = 3zy\partial_z - az^2x\partial_y, v_7 = -6yx\partial_y + az^2y\partial_z, \\ v_8 &= z^2y\partial_y, v_9 = z^2x\partial_x, v_{10} = z^2y\partial_z, v_{11} = z^2x\partial_z, v_{12} = zyx\partial_y, v_{13} = zyx\partial_x, \\ v_{14} &= zyx\partial_z, v_{15} = z^2yx\partial_y, v_{16} = z^2yx\partial_x, v_{17} = z^2yx\partial_z\end{aligned}$$

Degrees :

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
3	3	3	4	4	4	4	6	6	7	7	7	7	8	10	10	11

Solution space $H : H = \text{Span}\{1, z^2yx\}$

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